Slow relaxation in the large N model for phase ordering

Federico Corberi†, Eugenio Lippiello‡ and Marco Zannetti§

Istituto Nazionale per la Fisica della Materia, Unità di Salerno and Dipartimento di Fisica "E.Caianiello", Università di Salerno, 84081 Baronissi (Salerno), Italy

†corberi@na.infn.it ‡lippiello@sa.infn.it

§zannetti@na.infn.it

Abstract

The basic features of the slow relaxation phenomenology arising in phase ordering processes are obtained analytically in the large N model through the exact separation of the order parameter into the sum of thermal and condensation components. The aging contribution in the response function $\chi_{ag}(t,t_w)$ is found to obey a pattern of behavior, under variation of dimensionality, qualitatively similar to the one observed in Ising systems. There exists a critical dimensionality ($d=4$) above which $\chi_{ag}(t,t_w)$ is proportional to the defect density $\rho_D(t)$, while for $d<4$ it vanishes more slowly than $\rho_D(t)$ and at $d=2$ does not vanish. As in the Ising case, this behavior can be understood in terms of the dependence on dimensionality of the interplay between the defect density and the effective response associated to a single defect.

PACS: 05.70.Ln, 64.60.Cn, 05.40.-a, 05.50.+q

I. INTRODUCTION

The phase ordering processes [1] following the quench of non disorder systems (e.g. ferromagnets) below the critical point provide a simplified framework for the study of slow

relaxation phenomena. In particular, aging and the off equilibrium deviation from the fluctuation dissipation theorem (FDT) have been studied numerically [2–4] and through analytical treatment of solvable models [5–7]. The basic structure of the complex phenomenology arising in these processes originates from the wide separation in the time scales of fast and slow variables. Referring to the more intuitive case of a domain forming system, in the late stage of phase ordering the order parameter can be assumed to be the sum of two statistically independent components [8]

$$\phi(\vec{x},t) = \psi(\vec{x},t) + \sigma(\vec{x},t) \tag{1}$$

the first describing equilibrium thermal fluctuation within domains and the second off-equilibrium fluctuations due to interface motion. From (1) follows rather straightforwardly the split of the autocorrelation function

$$G(t,t_w) = G_{st}(t-t_w) + G_{ag}\left(\frac{t}{t_w}\right) \tag{2}$$

where $G_{st}(t-t_w)$ is the stationary time translation invariant (TTI) contribution due to ψ and $G_{ag}(t/t_w)$ the aging contribution due to the off-equilibrium σ degrees of freedom. A similar structure shows up in the linear response at the time t to an external random field switched on at the earlier time t_w

$$\chi(t,t_w) = \chi_{st}(t-t_w) + \chi_{ag}(t,t_w). \tag{3}$$

Here, the stationary contribution $\chi_{st}(t-t_w)$ satisfies the equilibrium FDT with respect to $G_{st}(t-t_w)$, while $\chi_{ag}(t,t_w)$ is the off-equilibrium extra response due to the presence of the interfaces. This is considered [2,6,3] to be proportional to the interface density

$$\chi_{ag}(t,t_w) \sim \rho_I(t) \tag{4}$$

where $\rho_I(t) \sim L^{-1}(t)$ and $L(t) \sim t^{1/z}$ is the typical domain size with $z=2$ for non conserved order parameter [1], as we shall assume in the following. A more precise formulation of this behavior is through the scaling relation

$$\chi_{ag}(t,t_w) = t_w^{-a}\hat{\chi}\left(\frac{t}{t_w}\right) \tag{5}$$

with the exponent $a = 1/2$. The implication is that $\chi_{ag}(t,t_w)$ is negligible in the asymptotic regime $t_w \to \infty$.

Motivated by analytical results [7,9] for the one dimensional Ising model which do not fit in the above scheme, recently we have undertaken a detailed study of the behavior of the response function under variation of space dimensionality for a system with a scalar order parameter [4]. On the basis of numerical simulations for discrete Ising spins and approximated analytical results for continuous spins, we have arrived to a picture for the behavior of $\chi_{ag}(t,t_w)$ which modifies considerably the one presented above. This is best understood by introducing the notion of the effective response due to a single interface and defined by

$$\chi_{ag}(t,t_w) = \rho_I(t)\chi_{eff}(t,t_w). \tag{6}$$

Then, if (4) were to hold, $\chi_{eff}(t,t_w)$ ought to be a constant. Instead, we have found that for an Ising system this is the case only for $d > 3$, while for $d < 3$ there is the power law growth

$$\chi_{eff}(t,t_w) \sim (t-t_w)^{\alpha} \tag{7}$$

with numerical values for the exponent compatible with $\alpha = (3-d)/4$. At $d = 3$ the power law is replaced by logarithmic growth, promoting $d = 3$ to the role of a critical dimensionality. This result is interesting for two reasons. The first concerns the mechanism of the response and the role of the off-equilibrium degrees of freedom. The power law (7) reveals that for $d < 3$ the aging component of the response does not originate trivially just from the polarization of interfacial spins, but a more complex phenomenon producing large scale optimization of the position of domains with respect to the external field is at work. The second regards the overall behavior of $\chi_{ag}(t,t_w)$. Putting together (6) and (7) the exponent a in (5) acquires a dependence on dimensionality

$$a = \begin{cases} \frac{1}{2}, & d > 3 \\ \frac{d-1}{4}, & d < 3 \end{cases} \tag{8}$$

showing that $\lim_{t_w \to \infty} \chi_{ag}(t, t_w)$ does not vanish as $d \to 1$. This indicates that $d = 1$ plays the role of a lower critical dimensionality, where the off equilibrium response becomes persistent. An interesting consequence of this phenomenon is that the connection between static and dynamic properties [10], which holds for $d > 1$, is invalidated at $d = 1$. It should be added that in order to have a phase ordering process in $d = 1$ thermal fluctuations within domains must be suppressed [4].

As stated previously, the picture summarized above has been established on the basis of a combination of exact results for the $d = 1$ Ising model, numerical results for Ising systems with $d > 1$ and approximate analytical results for continuous spin systems. It is, then, interesting to test how general is the picture. As a step in this direction, we have considered the large N model where exact analytical calculations can be carried out [11]. The slow relaxation properties of the large N or equivalent mean field models, arising in the quench at or below T_C have been analyzed before [5,12]. What we do here, however, goes beyond previous results since we manage to reproduce exactly and analytically the scenario outlined for Ising systems. We show that in the large N model one can make explicitly the separation (1) of the order parameter into the sum of two independent components which are responsible of the stationary and aging contributions in (2). Then, we carry out analytically the corresponding separation (3) of the response function. After introducing the notion of defect density for the large N model, we derive by analogy with (6) the effective response per defect and we find a behavior, as dimensionality is varied, qualitatively similar to the one established for scalar systems.

The paper is organized as follows. In Section II the large N model is defined and the main static properties are reviewed. In Section III the solution of the equation of motion is presented and the analytical form of the autocorrelation function is obtained in quenches at and below the critical point. In Section IV the splitting of the order parameter into

independent components satisfying the requirements above described is carried out. In Section V the behavior of the integrated response function is considered in relation to the off equilibrium deviation from the FDT. Finally, concluding remarks are made in Section VI.

II. MODEL AND STATIC PROPERTIES

We consider the purely relaxational dynamics of a system with a non conserved order parameter governed by the Langevin equation

$$\frac{\partial\vec{\phi}(\vec{x},t)}{\partial t} = -\frac{\delta\mathcal{H}[\vec{\phi}]}{\delta\vec{\phi}(\vec{x},t)} + \vec{\eta}(\vec{x},t) \tag{9}$$

where $\vec{\phi} = (\phi_1, ..., \phi_N)$ is an N-component vector, $\vec{\eta}(\vec{x},t)$ is a gaussian white noise with expectations

$$\begin{cases} \langle\vec{\eta}(\vec{x},t)\rangle = 0 \\ \langle\eta_\alpha(\vec{x},t)\eta_\beta(\vec{x}',t')\rangle = 2T\delta_{\alpha,\beta}\delta(\vec{x}-\vec{x}')\delta(t-t') \end{cases} \tag{10}$$

and T is the temperature of the thermal bath. In the dynamical process of interest the system is initially prepared in the infinite temperature equilibrium state with expectations

$$\begin{cases} \langle\vec{\phi}(\vec{x})\rangle = 0 \\ \langle\phi_\alpha(\vec{x})\phi_\beta(\vec{x}')\rangle = \Delta\delta_{\alpha,\beta}\delta(\vec{x}-\vec{x}') \end{cases} \tag{11}$$

and at the time $t = 0$ is quenched to a lower final temperature T_F. The hamiltonian is of the Ginzburg-Landau form

$$\mathcal{H}[\vec{\phi}] = \int_V d^dx \left[\frac{1}{2}(\nabla\vec{\phi})^2 + \frac{r}{2}\vec{\phi}^2 + \frac{g}{4N}(\vec{\phi}^2)^2\right] \tag{12}$$

where $r < 0$, $g > 0$ and V is the volume of the system. In the large N limit the equation of motion for the Fourier transform of the order parameter $\vec{\phi}(\vec{k}) = \int_V d^dx\vec{\phi}(\vec{x})\exp(i\vec{k}\cdot\vec{x})$ takes the linear form

$$\frac{\partial\vec{\phi}(\vec{k},t)}{\partial t} = -[k^2 + I(t)]\vec{\phi}(\vec{k},t) + \vec{\eta}(\vec{k},t) \tag{13}$$

where

$$\begin{cases} \langle \vec{\eta}(\vec{k},t)\rangle = 0 \\ \langle \eta_\alpha(\vec{k},t)\eta_\beta(\vec{k}',t')\rangle = 2T_F\delta_{\alpha,\beta}V\delta_{\vec{k}+\vec{k}',0}\delta(t-t') \end{cases} \tag{14}$$

and the function of time

$$I(t) = r + \frac{g}{N}\langle \vec{\phi}^2(\vec{x},t)\rangle \tag{15}$$

must be determined self-consistently, with the average on the right hand side taken both over thermal noise and initial condition. If the volume V is kept finite the system equilibrates in a finite time t_{eq} and the order parameter probability distribution reaches the Gibbs state

$$P_{eq}[\vec{\phi}(\vec{k})] = \frac{1}{Z}e^{-\frac{1}{2T_F V}\sum_{\vec{k}}(k^2+\xi^{-2})\vec{\phi}(\vec{k})\cdot\vec{\phi}(-\vec{k})} \tag{16}$$

where ξ is the correlation length defined by the equilibrium value of $I(t)$ through

$$\xi^{-2} = r + \frac{g}{N}\langle \vec{\phi}^2(\vec{x})\rangle_{eq} \tag{17}$$

with $\langle\cdot\rangle_{eq}$ standing for the average taken with (16).

In order to analyze the properties of $P_{eq}[\vec{\phi}(\vec{k})]$ it is necessary to extract from (17) the dependence of ξ^{-2} on T and V. Evaluating the average, the above equation yields

$$\xi^{-2} = r + \frac{g}{V}\sum_{\vec{k}}\frac{T_F}{k^2+\xi^{-2}}. \tag{18}$$

The solution of this equation is well known [13] and here we summarize the main features. Separating the $\vec{k}=0$ term under the sum, for very large volume we may rewrite

$$\xi^{-2} = r + gT_F B(\xi^{-2}) + g\frac{T_F}{V\xi^{-2}} \tag{19}$$

where

$$B(\xi^{-2}) = \lim_{V\to\infty}\frac{1}{V}\sum_{\vec{k}}\frac{1}{k^2+\xi^{-2}} = \int\frac{d^dk}{(2\pi)^d}\frac{e^{-\frac{k^2}{\Lambda^2}}}{k^2+\xi^{-2}} \tag{20}$$

regularizing the integral by introducing the high momentum cutoff Λ. The function $B(x)$ is a non negative monotonically decreasing function with the maximum value at $x=0$

$$B(0) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-\frac{k^2}{\Lambda^2}}}{k^2} = (4\pi)^{-\frac{d}{2}} \frac{2}{d-2} \Lambda^{d-2} \quad . \tag{21}$$

By graphical analysis one can easily show that (19) admits a finite solution for all T_F. However, there exists the critical value of the temperature T_C defined by

$$r + g T_C B(0) = 0 \tag{22}$$

such that for $T_F > T_C$ the solution is independent of the volume, while for $T_F \leq T_C$ it depends on the volume. Using

$$B(x) = (4\pi)^{-\frac{d}{2}} x^{\frac{d}{2}-1} e^{\frac{x}{\Lambda^2}} \Gamma\left(1 - \frac{d}{2}, \frac{x}{\Lambda^2}\right) \tag{23}$$

where $\Gamma(1-\frac{d}{2}, \frac{x}{\Lambda^2})$ is the incomplete gamma function, for $0 < \frac{T_F - T_C}{T_C} \ll 1$ one finds (Appendix I) $\xi \sim (\frac{T_F - T_C}{T_C})^{-\nu}$ where $\nu = 1/2$ for $d > 4$ and $\nu = 1/(d-2)$ for $d < 4$, with logarithmic corrections for $d = 4$. At T_C one has $\xi \sim V^\lambda$ with $\lambda = 1/4$ for $d > 4$ and $\lambda = 1/d$ for $d < 4$, again with logarithmic corrections for $d = 4$. Finally, below T_C one finds $\xi^2 = \frac{M^2 V}{T_F}$ where $M^2 = M_0^2 \left(\frac{T_C - T_F}{T_C}\right)$ and $M_0^2 = -r/g$.

Let us now see what are the implications for the equilibrium state. As (16) shows the individual Fourier components are independent random variables, gaussianly distributed with zero average. The variance is given by

$$\frac{1}{N} \langle \vec{\phi}(\vec{k}) \cdot \vec{\phi}(-\vec{k}) \rangle_{eq} = V C_{eq}(\vec{k}) \tag{24}$$

where

$$C_{eq}(\vec{k}) = \frac{T_F}{k^2 + \xi^{-2}} \tag{25}$$

is the equilibrium structure factor. For $T_F > T_C$, all $\vec{k}$ modes behave in the same way, with the variance growing linearly with the volume. For $T_F \leq T_C$, instead, ξ^{-2} is negligible with respect to k^2 except at $\vec{k} = 0$, yielding

$$C_{eq}(\vec{k}) = \begin{cases} \frac{T_C}{k^2}(1 - \delta_{\vec{k},0}) + a V^{2\lambda} \delta_{\vec{k},0} & \text{, for } T_F = T_c \\ \frac{T_F}{k^2}(1 - \delta_{\vec{k},0}) + M^2 V \delta_{\vec{k},0} & \text{, for } T_F < T_c \end{cases} \tag{26}$$

where a is a constant (Appendix I). This produces a volume dependence in the variance of the $\vec{k}=0$ mode growing faster than linear. Therefore, for $T_F \leq T_C$ the $\vec{k}=0$ mode behaves differently from all the other modes with $\vec{k} \neq 0$. For $T_F < T_C$ the probability distribution (16) takes the form

$$P_{eq}[\vec{\phi}(\vec{k})] = \frac{1}{Z} e^{-\frac{\vec{\phi}^2(0)}{2M^2V^2}} e^{-\frac{1}{2T_F V}\sum_{\vec{k}} k^2 \vec{\phi}(\vec{k})\cdot\vec{\phi}(-\vec{k})} \quad . \tag{27}$$

Therefore, crossing T_C there is a transition from the usual disordered high temperature phase to a low temperature phase characterized by a macroscopic variance in the distribution of the $\vec{k}=0$ mode. The distinction between this phase and the mixture of pure states, obtained below T_C when N is kept finite, has been discussed elsewhere [14].

We shall refer to this transition as condensation of fluctuations in the $\vec{k}=0$ mode. In order to gain a better insight it is convenient to go back to real space, splitting the order parameter into the sum of the two independent components

$$\vec{\phi}(\vec{x}) = \vec{\sigma} + \vec{\psi}(\vec{x}) \tag{28}$$

with

$$\vec{\sigma} = \frac{1}{V}\vec{\phi}(\vec{k}=0) \tag{29}$$

and

$$\vec{\psi}(\vec{x}) = \frac{1}{V}\sum_{\vec{k}\neq 0} \vec{\phi}(\vec{k}) e^{i\vec{k}\cdot\vec{x}}. \tag{30}$$

Then (27) takes the form

$$P_{eq}[\vec{\phi}(\vec{x})] = P(\vec{\sigma})P[\vec{\psi}(\vec{x})] \tag{31}$$

where

$$P(\vec{\sigma}) = \frac{1}{(2\pi M^2)^{N/2}} e^{-\frac{\vec{\sigma}^2}{2M^2}} \tag{32}$$

shows the formation of the condensate below T_C with the macroscopic variance $\frac{1}{N}\langle\vec{\sigma}^2\rangle_{eq} = M^2$ while

$$P[\vec{\psi}(\vec{x})] = \frac{1}{Z} e^{-\frac{1}{2T_F}\int_V d^dx (\nabla\vec{\psi})^2} \tag{33}$$

describes thermal fluctuations about the condensate. Correspondingly, the correlation function $G_{eq}(\vec{x}-\vec{x}') = (1/N)\langle\vec{\phi}(\vec{x})\cdot\vec{\phi}(\vec{x}')\rangle_{eq}$ splits into the sum of two pieces

$$G_{eq}(\vec{x}-\vec{x}') = G_T(\vec{x}-\vec{x}') + M^2 \tag{34}$$

where

$$G_T(\vec{x}-\vec{x}') = \frac{1}{N}\langle\vec{\psi}(\vec{x})\cdot\vec{\psi}(\vec{x}')\rangle_{eq} \tag{35}$$

is the correlation of thermal fluctuations and M^2 comes from fluctuations of the condensate. Since $G_T(\vec{x}-\vec{x}')$ at large distances decays like $|\vec{x}-\vec{x}'|^{2-d}$, from (34) follows $\lim_{|\vec{x}-\vec{x}'|\to\infty} G_{eq}(\vec{x}-\vec{x}') = M^2$ showing the violation of the clustering property of the correlation function due to the breaking of ergodicity in the low temperature phase. For future reference, notice that from (17) follows that for $\vec{x}=\vec{x}'$ and $T_F \leq T_C$

$$G_{eq}(0) = M_0^2 \tag{36}$$

which in turn implies

$$G_T(0) = M_0^2 - M^2. \tag{37}$$

As stated above, with a finite V equilibrium is reached for $t \sim t_{eq}$ and $t_{eq} \sim \xi^2$. Hence, in a quench to $T_F < T_C$ one has $t_{eq} \sim V$ implying that if the $V \to \infty$ limit is taken at the beginning of the quench, equilibrium is not reached for any finite time. Since in the following we are interested in the relaxation regime before equilibrium is reached, we shall take the thermodynamic limit from the outset. We are interested to see whether it is possible to carry out the decomposition (1) of the order parameter in such a way that $\vec{\sigma}(\vec{x},t)$ eventually evolves into the equilibrium condensate $\vec{\sigma}$ and $\vec{\psi}(\vec{x},t)$ into the equilibrium thermal fluctuations.

III. DYNAMICS

Due to rotational symmetry, from now on we shall drop vectors and refer to the generic component of the order parameter. The formal solution of (13) is given by

$$\phi(\vec{k},t) = R(\vec{k},t,0)\phi_0(\vec{k}) + \int_0^t dt' R(\vec{k},t,t')\eta(\vec{k},t') \tag{38}$$

where $\phi_0(\vec{k}) = \phi(\vec{k},t=0)$ and according to (11) in the $V \to \infty$ limit

$$\begin{cases} \langle \phi_0(\vec{k}) \rangle = 0 \\ \langle \phi_0(\vec{k})\phi_0(\vec{k}') \rangle = \Delta(2\pi)^d \delta(\vec{k}+\vec{k}'). \end{cases} \tag{39}$$

The response function is given by

$$R(\vec{k},t,t') = \frac{Y(t')}{Y(t)} e^{-k^2(t-t')} \tag{40}$$

with $Y(t) = \exp[Q(t)]$, $Q(t) = \int_0^t ds I(s)$ and $Y(0) = 1$. The actual solution is obtained once the function $Y(t)$ is determined. In order to do this, notice that from the definition of $Y(t)$ follows

$$\frac{dY^2(t)}{dt} = 2\left[r + g\langle \phi^2(\vec{x},t) \rangle\right] Y^2(t). \tag{41}$$

Writing $\langle \phi^2(\vec{x},t) \rangle$ in terms of the structure factor

$$\langle \phi^2(\vec{x},t) \rangle = \int \frac{d^dk}{(2\pi)^d} C(\vec{k},t) e^{-\frac{k^2}{\Lambda^2}} \tag{42}$$

and using (38) to evaluate $C(\vec{k},t)$

$$C(\vec{k},t) = R^2(k,t,0)\Delta + 2T_F \int_0^t dt' R^2(\vec{k},t,t') \tag{43}$$

from (41) we obtain the integro-differential equation

$$\frac{dY^2(t)}{dt} = 2rY^2(t) + 2g\Delta f\left(t + \frac{1}{2\Lambda^2}\right) + 4gT_F \int_0^t dt' f\left(t - t' + \frac{1}{2\Lambda^2}\right) Y^2(t') \tag{44}$$

where

$$f(x) \equiv \int \frac{d^dk}{(2\pi)^d} e^{-2k^2x} = (8\pi x)^{-\frac{d}{2}}. \tag{45}$$

Solving (44) by Laplace transform [15,12], the leading behavior of $Y(t)$ for large time is given by

$$Y^2(t) = \begin{cases} A_a e^{2\xi^{-2}t} & ; \text{for } T_F > T_C \\ A_c t^{\omega} & ; \text{for } T_F = T_C \\ A_b t^{-\frac{d}{2}} & ; \text{for } T_F < T_C \end{cases} \tag{46}$$

where $\omega = 0$ for $d > 4$ and $\omega = (d-4)/2$ for $d < 4$. The values of the constants A_a, A_b and A_c are listed in Appendix II.

This completes the solution of the model. Once the response function is known, we may go back to (38) and take various averages. Using (14) and (39) we have

$$\langle \phi(\vec{k}, t) \rangle = 0 \tag{47}$$

and for the two time structure factor $\langle \phi(\vec{k}, t)\phi(\vec{k}', t')\rangle = C(\vec{k}, t, t')(2\pi)^d \delta(\vec{k} + \vec{k}')$ with $t \geq t'$

$$C(\vec{k}, t, t') = R(\vec{k}, t, 0)R(\vec{k}, t', 0)\Delta + 2T_F \int_0^{t'} dt'' R(\vec{k}, t, t'')R(\vec{k}, t', t''). \tag{48}$$

The corresponding real space correlation function $G(\vec{x} - \vec{x}', t, t') = \langle \phi(\vec{x}, t)\phi(\vec{x}', t')\rangle$ is given by

$$\begin{aligned} G(\vec{x} - \vec{x}', t, t') = & \int d\vec{x}'' R(\vec{x} - \vec{x}'', t, 0)R(\vec{x}' - \vec{x}'', t', 0)\Delta \\ & + 2T_F \int_0^{t'} dt'' \int d\vec{x}'' R(\vec{x} - \vec{x}'', t, t'')R(\vec{x}' - \vec{x}'', t', t'') \end{aligned} \tag{49}$$

where $R(\vec{x}, t, t')$ is the inverse Fourier transform of $R(\vec{k}, t, t')$. In the following we will be primarily concerned with the autocorrelation function $G(t, t') = G(\vec{x} - \vec{x}' = 0, t, t')$. Using the definitions (40) and (45) this is given by

$$G(t, t') = \frac{1}{Y(t)Y(t')} \left[f\left(\frac{t + t'}{2} + \frac{1}{2\Lambda^2}\right)\Delta + 2T_F \int_0^{t'} dt'' f\left(\frac{t + t'}{2} - t'' + \frac{1}{2\Lambda^2}\right) Y^2(t'') \right]. \tag{50}$$

The behavior of this quantity for different final temperatures and for different time regimes has been studied in the literature [15,12]. Here we summarize the results.

For $T_F > T_C$ from (46) follows that for $t' > t_{eq} = 2\xi^{-2}$ the autocorrelation function is TTI

$$G(\tau) = G_{eq}(0) e^{-\frac{\tau}{t_{eq}}} \tag{51}$$

where $G_{eq}(0)$ is the equilibrium fluctuation above T_C given by (17), i.e. $G_{eq}(0) = M_0^2 + \frac{1}{g}\xi^{-2}$ and $\tau = t - t'$.

For $T_F \leq T_c$ the equilibration time diverges and there are two time regimes of interest:

i) short time separation: $t' \to \infty \quad , \quad \frac{\tau}{t'} \to 0$

ii) large time separation: $t' \to \infty \quad , \quad \frac{\tau}{t'} \to \infty$.

Taking t' large and using (46), for $T_F = T_C$ in these limits we find

$$G(t,t') = \begin{cases} M_0^2(\Lambda^2\tau + 1)^{1-\frac{d}{2}} & , \text{for } \frac{\tau}{t'} \to 0 \\ A t'^{1-\frac{d}{2}} F(\frac{t}{t'}) & , \text{for } \frac{\tau}{t'} \to \infty \end{cases} \tag{52}$$

with

$$F(x) = \begin{cases} \frac{4}{(4\pi)^{d/2}(d-2)}(x-1)^{1-d/2}\frac{x^{1-d/4}}{x+1} & , \text{for } 2 < d < 4 \\ \frac{4}{(8\pi)^{d/2}(d-2)}\left[\left(\frac{x-1}{2}\right)^{1-d/2} - \left(\frac{x+1}{2}\right)^{1-d/2}\right] & , \text{for } d > 4 \end{cases} \tag{53}$$

and for $T_F < T_c$

$$G(t,t') = \begin{cases} M^2 + (M_0^2 - M^2)(\Lambda^2\tau + 1)^{1-\frac{d}{2}} & , \text{for } \frac{\tau}{t'} \to 0 \\ M^2\left[\frac{4tt'}{(t+t')^2}\right]^{\frac{d}{4}} & , \text{for } \frac{\tau}{t'} \to \infty. \end{cases} \tag{54}$$

In the latter case we may also write

$$G(t,t') = G_{st}(\tau) + G_{ag}\left(\frac{t'}{t}\right) \tag{55}$$

where

$$G_{st}(\tau) = (M_0^2 - M^2)(\Lambda^2\tau + 1)^{1-\frac{d}{2}} \tag{56}$$

and

$$G_{ag}(x) = M^2\left[\frac{4x}{(1+x)^2}\right]^{\frac{d}{4}}. \tag{57}$$

This is illustrated in Fig.1 which shows the convergence toward the form (55) of the exact autocorrelation function $G(t,t')$ obtained by solving numerically the coupled set of equations (44) and (50).

As explained in the Introduction, this behavior is suggestive of the existence of two variables responsible respectively of the stationary and of the aging behaviors. We want now to show that in the large N limit these two variables can be explicitly constructed.

IV. SPLITTING OF THE FIELD

The task stated at the end of the previous Section requires the splitting of the order parameter field into the sum of two independent contributions

$$\phi(\vec{x},t) = \psi(\vec{x},t) + \sigma(\vec{x},t) \tag{58}$$

with zero averages $\langle \psi(\vec{x},t) \rangle = \overline{\sigma(\vec{x},t)} = 0$ and autocorrelation functions such that

$$\langle \psi(\vec{x},t)\psi(\vec{x},t') \rangle = G_{st}(\tau) \tag{59}$$

$$\overline{\sigma(\vec{x},t)\sigma(\vec{x},t')} = G_{ag}(t'/t). \tag{60}$$

In order to stress the statistical independence of the two component fields, we have used the angular brackets for averages over ψ and the overbar for averages over σ.

For the construction of fields with these properties, let us go back to Fourier space. Using the multiplicative property of the response function

$$R(\vec{k},t,t')R(\vec{k},t',t_0) = R(\vec{k},t,t_0) \tag{61}$$

with $t > t' > t_0$ it is easy to show that the formal solution (38) of the equation of motion can be rewritten as the sum of two statistically independent components $\phi(\vec{k},t) = \psi(\vec{k},t) + \sigma(\vec{k},t)$ with

$$\sigma(\vec{k},t) = R(\vec{k},t,t_0)\phi(\vec{k},t_0) \tag{62}$$

and

$$\psi(\vec{k},t) = \int_{t_0}^{t} dt' R(\vec{k},t,t')\eta(\vec{k},t') \tag{63}$$

since for $0 \leq t_0 < t$, $\phi(\vec{k}, t_0)$ and $\eta(\vec{k}, t)$ are independent by causality. In other words, the order parameter at the time t is split into the sum of a component $\sigma(\vec{k}, t)$ driven by the fluctuations of the order parameter at the earlier time t_0 and a component $\psi(\vec{k}, t)$ driven by the thermal history between t_0 and t. Let us remark that t_0 can be chosen arbitrarily between the initial time of the quench ($t = 0$) and the observation time t. With the particular choice $t_0 = 0$, the component $\sigma(\vec{k}, t)$ is driven by the fluctuations in the initial condition (39). The ψ component describes fluctuations of thermal origin while the σ component, as it will be clear below, if t_0 is chosen sufficiently large, describes the local condensation of the order parameter.

According to definitions (62) and (63), from (14) and (39) follows $\overline{\sigma(\vec{k}, t)} = \langle \psi(\vec{k}, t) \rangle = 0$. The two time structure factor splits into the sum

$$C(\vec{k}, t, t') = C_\sigma(\vec{k}, t, t') + C_\psi(\vec{k}, t, t') \tag{64}$$

with $\overline{\sigma(\vec{k}, t)\sigma(\vec{k}', t')} = C_\sigma(\vec{k}, t, t')(2\pi)^d \delta(\vec{k} + \vec{k}')$ and

$$C_\sigma(\vec{k}, t, t') = R(\vec{k}, t, t_0) R(\vec{k}, t', t_0) C(\vec{k}, t_0) \tag{65}$$

where $C(\vec{k}, t_0)$ is the equal time structure factor at the time t_0. Similarly $\langle \psi(\vec{k}, t)\psi(\vec{k}', t') \rangle = C_\psi(\vec{k}, t, t')(2\pi)^d \delta(\vec{k} + \vec{k}')$ with

$$C_\psi(\vec{k}, t, t') = 2T_F \int_{t_0}^{t'} dt'' R(\vec{k}, t, t'') R(\vec{k}, t', t''). \tag{66}$$

Going to real space and setting $\vec{x} = \vec{x}'$, we have

$$G(t, t') = G_\sigma(t, t') + G_\psi(t, t') \tag{67}$$

with

$$G_\sigma(t, t') = \frac{Y^2(t_0)}{Y(t)Y(t')} \int \frac{d^d k}{(2\pi)^d} e^{-k^2\left(t + t' - 2t_0 + \frac{1}{\Lambda^2}\right)} C(\vec{k}, t_0) \tag{68}$$

and

$$G_\psi(t, t') = 2T_F \int_{t_0}^{t'} dt'' \frac{Y^2(t'')}{Y(t)Y(t')} \int \frac{d^d k}{(2\pi)^d} e^{-k^2\left(t + t' - 2t'' + \frac{1}{\Lambda^2}\right)}. \tag{69}$$

Assuming that t and t' are sufficiently larger than t_0 so that $C(\vec{k}, t_0)$ under the integral in (68) can be replaced by its value at $\vec{k} = 0$ and using (45) we may write

$$G_\sigma(t, t') = \frac{Y^2(t_0)}{Y(t)Y(t')} f\left(\frac{t+t'}{2} - t_0 + \frac{1}{2\Lambda^2}\right) C_0 \tag{70}$$

$$G_\psi(t, t') = \frac{2T_F}{Y(t)Y(t')} \int_{t_0}^{t'} dt'' f\left(\frac{t+t'}{2} - t'' + \frac{1}{2\Lambda^2}\right) Y^2(t'') \tag{71}$$

where $C_0 = C(\vec{k} = 0, t_0)$.

Let us now evaluate these results for short and large time separations. In the first case the dominant contributions are given by

$$G_\sigma(t, t') = M^2 - (M_0^2 - M^2)(2\Lambda^2 t_0)^{1-\frac{d}{2}} \tag{72}$$

and

$$G_\psi(t, t') = (M_0^2 - M^2)\left[(2\Lambda^2 t_0)^{1-\frac{d}{2}} + (\Lambda^2\tau + 1)^{1-\frac{d}{2}}\right] \tag{73}$$

where the unknown constant C_0 entering (70) has been eliminated imposing that the equilibrium sum rule $G_\sigma(0) + G_\psi(0) = M_0^2$ be satisfied. The sum of the above contributions is independent of t_0, as it should, and coincides with (54). Similarly, in the large time regime we find the t_0 dependent results

$$G_\sigma(t, t') = \left[M^2 - (M_0^2 - M^2)(2\Lambda^2 t_0)^{1-\frac{d}{2}}\right]\left[\frac{4tt'}{(t+t')^2}\right]^{\frac{d}{4}} \tag{74}$$

and

$$G_\psi(t, t') = (M_0^2 - M^2)(2\Lambda^2 t_0)^{1-\frac{d}{2}}\left[\frac{4tt'}{(t+t')^2}\right]^{\frac{d}{4}} \tag{75}$$

again with the sum independent of t_0 and giving back (54). Defining the microscopic time $t^* = \Lambda^{-2}$, we see that taking $t_0 \gg t^*$ from (72) and (73) we get the short time behavior

$$\begin{cases} G_\sigma(t, t') = M^2 \\ G_\psi(t, t') = (M_0^2 - M^2)(\Lambda^2\tau + 1)^{1-\frac{d}{2}} \end{cases} \tag{76}$$

while from (74) and (75) follows the large time behavior

$$\begin{cases} G_\sigma(t,t') = M^2 \left[\frac{4tt'}{(t+t')^2}\right]^{\frac{d}{4}} \\ G_\psi(t,t') = 0. \end{cases} \tag{77}$$

Comparing with (56) and (57) we can make the identifications

$$G_\psi(t,t') = G_{st}(\tau) \tag{78}$$

and

$$G_\sigma(t,t') = G_{ag}\left(\frac{t'}{t}\right). \tag{79}$$

Therefore, the fields $\sigma(\vec{x},t)$ and $\psi(\vec{x},t)$ defined by (62) and (63), with the choice of $t_0 \gg t^*$, provide an explicit realization of the decomposition (58) satisfying the requirements (59) and (60). The physical meaning of the two components can be readily understood comparing the equal time values of (76), which yield respectively $G_\sigma(t=t') = M^2$ and $G_\psi(t=t') = M_0^2 - M^2$, with the equilibrium results (36) and (37). It is then clear that the field $\sigma(\vec{x},t)$ is associated to local condensation of the order parameter with fluctuations of size M^2, while the field $\psi(\vec{x},t)$ describes thermal fluctuations. In the case of a system with a scalar order parameter, σ would be associated to the average value of the order parameter characteristic of a domain and ψ to thermal fluctuations within domains [4]. This makes also clear the origin of the requirement $t_0 \gg t^*$. In order to make a separation of variables with the above physical meaning, it is necessary to wait a time t_0 large enough for all microscopic transients to have occurred leaving well formed local equilibrium.

V. RESPONSE FUNCTION

In the previous Section we have produced the explicit separation of the order parameter into the condensation component and thermal fluctuations in the quench at $T_F < T_C$. It is now interesting to see in what relation these components are with the linear response function.

If an external field is switched on at the time $t_w > t_0$, the splitting (58) modifies into $\vec{\phi}_h(\vec{x},t) = \vec{\psi}(\vec{x},t) + \vec{\sigma}_h(\vec{x},t)$ where to linear order

$$\vec{\sigma}_h(\vec{x},t) = \vec{\sigma}(\vec{x},t) + \int_{t_w}^{t} dt' \int_V d\vec{x}' R(\vec{x}-\vec{x}',t,t')\vec{h}(\vec{x}') \tag{80}$$

with $\sigma(\vec{x},t), \psi(\vec{x},t)$ and $R(\vec{x},t,t')$ unperturbed quantities. Here, we are interested in the response to a quenched, gaussianly distributed random field with expectations

$$\begin{cases} E_h[\vec{h}(\vec{x})] = 0 \\ E_h[h_\alpha(\vec{x})h_\beta(\vec{x}')] = h_0^2 \delta_{\alpha\beta}\delta(\vec{x}-\vec{x}'). \end{cases} \tag{81}$$

Computing the staggered magnetization from (80) and averaging over the field we obtain

$$\frac{1}{N h_0^2 V} \int_V d\vec{x} E_h[\overline{\vec{\sigma}_h(\vec{x},t)} \cdot \vec{h}(\vec{x})] = \chi(t,t_w) \tag{82}$$

where $\chi(t,t_w) = \int_{t_w}^{t} R(t,t')$ is the integrated response function and $R(t,t') = R(\vec{x}-\vec{x}' = 0,t,t')$. From (40) and (46) for $T_F < T_c$

$$R(t,t') = \int \frac{d^d k}{(2\pi)^d} R(\vec{k},t,t') e^{-\frac{k^2}{\Lambda^2}} = (4\pi)^{-\frac{d}{2}} \left(\frac{t'}{t}\right)^{-\frac{d}{4}} \left(t - t' + \frac{1}{\Lambda^2}\right)^{-\frac{d}{2}}. \tag{83}$$

Let us then write the integrated response function as the sum

$$\chi(t,t_w) = \chi_{st}(t-t_w) + \chi_{ag}(t,t_w) \tag{84}$$

where the stationary component $\chi_{st}(t-t_w)$ is defined by requiring that the equilibrium FDT be satisfied with respect to the stationary component (78) of the autocorrelation function, namely

$$T_F \chi_{st}(t-t_w) = G_\psi(0) - G_\psi(t-t_w). \tag{85}$$

It is straightforward to check that this is verified by

$$\begin{aligned} \chi_{st}(t-t_w) &= (4\pi)^{-\frac{d}{2}} \int_{t_w}^{t} dt' \left(t - t' + \frac{1}{\Lambda^2}\right)^{-\frac{d}{2}} \\ &= \frac{1}{T_F}(M_0^2 - M^2)\left\{1 - [\Lambda^2(t-t_w)+1]^{1-\frac{d}{2}}\right\}. \end{aligned} \tag{86}$$

The aging component then remains defined by the difference

$$
\begin{aligned}
\chi_{ag}(t,t_w) &= \chi(t,t_w) - \chi_{st}(t-t_w) \\
&= (4\pi)^{-\frac{d}{2}} t_w^{1-\frac{d}{2}} x^{\frac{d}{2}-1} \int_x^1 dy \left(1 - y + \frac{x}{\Lambda^2 t_w}\right)^{-\frac{d}{2}} \left(y^{-\frac{d}{4}} - 1\right)
\end{aligned} \tag{87}
$$

where $x = t_w/t$. This shows that for $d > 2$ and any fixed value of x $\lim_{t_w \to \infty} \chi_{ag}(t,t_w) = 0$ implying

$$
\lim_{t_w \to \infty} \chi(t,t_w) = \chi_{st}(t-t_w). \tag{88}
$$

Hence, in the limit $t_w \to \infty$ the equilibrium FDT (85) is satisfied by the whole response function and the plot of $\chi(t,t_w)$ vs $G_\psi(t,t_w)$ is linear. Instead, if $\chi(t,t_w)$ is plotted against the full autocorrelation function (67), from (55) follows

$$
\lim_{t_w \to \infty} T_F \chi(t,t_w) = \begin{cases} G(t,t) - G(t,t_w) & , \text{for } M^2 < G(t,t_w) \leq M_0^2 \\ M_0^2 - M^2 & , \text{for } G(t,t_w) < M^2 \end{cases} \tag{89}
$$

yielding the behavior of Fig.2 which is characteristic of the phase ordering process [2,3].

For $d = 2$ the power of t_w in front of the integral disappairs from (87) and the leading contribution for $t \gg \Lambda^{-2}$ is given by

$$
\chi_{ag}(t,t_w) = \frac{1}{2\pi} \log \left(\frac{2}{1+\sqrt{x}} \right) \tag{90}
$$

showing that the aging contribution to the response does not vanish as $t_w \to \infty$. Furthermore, since $T_C = 0$ for $d = 2$, the phase-ordering process requires $T_F = 0$ and from (67) follows

$$
G(t,t') = G_\sigma(t,t') = 2M_0^2 \frac{\sqrt{x}}{1+x}. \tag{91}
$$

Eliminating x between (90) and (91) we get

$$
\chi_{ag}(G) = \frac{1}{2\pi} \log \left\{ \frac{2}{1 + \frac{M_0^2}{G}\left[1 - \sqrt{1 - \frac{G^2}{M_0^4}}\right]} \right\} \tag{92}
$$

which gives a parametric plot (Fig. 3) qualitatively similar to what one finds in the Ising model for $d = 1$.

As it was recalled in the Introduction, with a scalar order parameter the behavior of $\chi_{ag}(t, t_w)$ under variations of dimensionality becomes trasparent by introducing the effective response $\chi_{eff}(t, t_w)$ associated to a single interface. The mechanism regulating the behavior of $\chi_{ag}(t, t_w)$ then is explained through the balance between the rate of loss of interfaces as coarsening proceeds, and the rate of growth of the single interface response given by (7). The dimensionality dependence of the exponent α in that case is the outcome [4] of the competition between the external field and the curvature of interfaces in the drive of interface motion.

In the large N model there are no localized defects and none of the above concepts has direct physical meaning. Nonetheless, the notion of defect density can be extended to the large N case by looking at the behavior of $I(t)$. Consider the quench to $T_F = 0$ where there are no thermal fluctuations. From (46), for large time, $I(t) \sim -t^{-1}$ namely

$$M_0^2 - \frac{1}{N}\langle \vec{\phi}^2(\vec{x}, t)\rangle \sim t^{-1}. \tag{93}$$

If the system was in the ordered state, with the order parameter aligned everywhere, the left hand side ought to vanish. Therefore, the positive difference (93) may be attributed to "defects" with density

$$\rho_D(t) \sim L^{-2}(t) \sim t^{-1} \tag{94}$$

which is what one obtains in general with a vector order parameter [1]. We may then define the effective response function per defect by the analogue of (6) $\chi_{ag}(t, t_w) = \rho_D(t)\chi_{eff}(t, t_w)$ which yields

$$\chi_{eff}(t, t_w) = t^{2-\frac{d}{2}} \int_x^1 dy \frac{y^{-\frac{d}{4}} - 1}{\left(1 - y + \frac{1}{\Lambda^2 t}\right)^{\frac{d}{2}}}. \tag{95}$$

The behavior of this quantity can be computed analytically for short and for large time separation obtaining in both cases, apart from a change in the prefactor, a power law behavior as in (6)

$$\chi_{eff}(t, t_w) \sim (t - t_w)^{\alpha} \tag{96}$$

with

$$\alpha = \begin{cases} 0 & , \text{for } d > 4 \\ 2 - \frac{d}{2} & , \text{for } d < 4 \end{cases} \tag{97}$$

and

$$\chi_{eff}(t, t_w) \sim \log\left[\Lambda^2(t - t_w)\right] \tag{98}$$

for $d = 4$. The overall exact behavior of $\chi_{eff}(t, t_w)$ is depicted in Fig.4, obtained by plotting (95) for different dimensionalities.

Therefore, the qualitative picture is the same obtained in the scalar case [4]. There exist upper critical dimensionalities, $d_U = 3$ for $N = 1$ and $d_U = 4$ for $N = \infty$ (presumably for all $N > 1$), above which χ_{eff} saturates to a constant value within microscopic times. At d_U χ_{eff} grows logarithmically, while below d_U there is power law growth. In addition there are lower critical dimensionalities, $d_L = 1$ for $N = 1$ and $d_L = 2$ for $N = \infty$ (or $N > 1$), where the exponent α reaches exactly the value such that $\chi_{eff}(t, t_w) \sim \rho_I^{-1}(t)$ in the scalar case and $\chi_{eff}(t, t_w) \sim \rho_D^{-1}(t)$ in the vectorial case. Namely, at d_L the growth of χ_{eff} makes up exactly for the loss of interfaces or defects, producing the behavior illustrated in Fig. 3.

VI. CONCLUDING REMARKS

In this paper we have shown that the basic features of the slow relaxation phenomenology arising in phase ordering processes: separation of order parameter, autocorrelation function and linear response function into fast and slow components, can be obtained analytically and exactly in the large N model. The behavior of $\chi_{ag}(t, t_w)$ is of particular interest since it displays the same qualitative pattern of behavior under variation of dimensionality observed in the Ising case. This is a strong indication that this might be a generic feature of the slow relaxation in phase ordering processes. In this respect it might be worth to undertake a numerical study of $\chi_{ag}(t, t_w)$ in systems with vector order parameter and finite N at different dimensionalities.

A comment should be made about the connection between static and dynamic properties. One of the most interesting recent developments in the study of the off equilibrium deviation from FDT has been the derivation [10] of a link between the response function and the structure of the equilibrium state. Assuming that $\lim_{t_w \to \infty} \chi(t, t_w) = \chi(G(t, t_w))$, i.e. that in the large time regime $\chi(t, t_w)$ depends on time only through the autocorrelation function, this connection takes the form

$$P(q) = -T_F \left. \frac{d^2 \chi(G)}{dG^2} \right)_{G=q} \tag{99}$$

where $P(q)$ is the probability that in the equilibrium state the overlap $\frac{1}{NV} \int_V d\vec{x} \vec{\phi}(\vec{x}) \cdot \vec{\phi}'(\vec{x})$ between two different configurations $[\vec{\phi}(\vec{x})]$ and $[\vec{\phi}'(\vec{x})]$ takes the value q. For Ising systems (99) holds for $d > 1$. In fact, for $d > 1$ the $t_w \to \infty$ limit of $\chi(t, t_w)$ is found [2,6,4] to have the form (89) which, applying (99), yields

$$P(q) = \delta(q - M^2) \tag{100}$$

in agreement with an equilibrium state formed by the mixture of pure states. For $d = 1$, since $\chi_{ag}(t, t_w)$ does not disappear as $t_w \to \infty$, the connection (99) is no more valid [4]. Then, in the large N model we should expect (99) to fail at most for $d = 2$, since that is the case where $\chi_{ag}(t, t_w)$ does not asymptotically vanish. However, it is not so straightforward that (99) should hold for $d > 2$. In fact, as observed above, the behavior of $\chi(t, t_w)$ in the limit $t_w \to \infty$ for $d > 2$ is indistinguishable from what one obtains in the Ising case. Namely, one finds the form (89) of $\chi(G)$ which yields (100) for $P(q)$ and this is what one expects when there is a mixture of ordered pure states. The problem is, as explained in Section 2, that the structure of the low temperature state in the large N model is quite different from a mixture of pure states. This puzzling feature of the large N model will be investigated in a separate paper.

Acknowledgments - This work has been partially supported from the European TMR Network-Fractals c.n. FMRXCT980183 and from MURST through PRIN-2000.

VII. APPENDIX I

Solving (19) in the large volume limit one finds:

for $0 < \frac{T_F - T_C}{T_C} \ll 1$

$$(\xi\Lambda)^{-2} = \begin{cases} \frac{\Lambda^2}{2}\left[\frac{M_0^2}{2T_F}\frac{(8\pi)^{d/2}}{\Gamma(1-d/2)}\right]^{-\frac{2}{2-d}} & \text{, for } d < 2 \\ e^{-\frac{4\pi M_0^2}{T_F}} & \text{, for } d = 2 \\ \left[\frac{T_F - T_C}{T_C}\frac{1}{\Gamma(2-d/2)}\right]^{\frac{2}{d-2}} & \text{, for } 2 < d < 4 \\ \frac{T_F - T_C}{T_C}\left(\frac{\Lambda^2}{M_0^2 g} + \frac{T_F}{T_C}\frac{2}{d-4}\right)^{-1} & \text{, for } d > 4 \end{cases} \tag{101}$$

and

$$(\xi\Lambda)^{-2}\log\left((\xi\Lambda)^{-2}\right) = -\left(\frac{T_F - T_C}{T_F}\right) \quad \text{, for } d = 4. \tag{102}$$

For $T_F = T_C$

$$\xi = \begin{cases} \frac{M_0^2}{T_C}\left(\frac{d-2}{d-4} - \frac{\Lambda^2}{r}\right) V^{\frac{1}{4}} & \text{, for } d > 4 \\ \left[\Gamma(\frac{4-d}{2}) T_C^{\frac{d}{2}-1} M_0^2 \Lambda^{2-d}\right]^{\frac{1}{d}} V^{\frac{1}{d}} & \text{, for } d < 4 \end{cases} \tag{103}$$

and

$$\xi[2\log(\xi\Lambda)]^{-\frac{1}{4}} = \left(\frac{M_0^2}{\Lambda^2 T_C} V\right)^{\frac{1}{4}} \tag{104}$$

for $d = 4$.

VIII. APPENDIX II

The prefactors in (46) are given by

$$A_a = \begin{cases} \Delta\left[\frac{M_0^2}{\Gamma(1-d/2)}\left(\frac{8\pi}{T_F}\right)^{d/2}\right]^{-\frac{2}{2-d}} & \text{, for } d < 2 \\ 2\pi\frac{\Delta}{T_F^2} M_0^2 e^{-4\pi\frac{M_0^2}{T_F}} & \text{, for } d = 2 \\ \left[\frac{M_0^2}{T_C}(T_F - T_C)\right]^{\frac{4-d}{2-d}}\left(\frac{1}{2g} + \frac{\Delta M_0^2}{2T_C}\right)\left[\frac{2T_F|\Gamma(1-d/2)|}{(8\pi)^{d/2}}\right]^{\frac{2}{d-2}}\frac{1}{d-2} & \text{, for } 2 < d < 4 \\ -\frac{(8\pi)^2}{2T_C}\left(\frac{1}{2g} + \frac{\Delta M_0^2}{2T_C}\right)\frac{l}{\log(2\xi^{-2})} & \text{, for } d = 4 \\ \left(\frac{1}{2g} + \frac{\Delta M_0^2}{2T_c}\right)\left(\frac{1}{2g} + \frac{M_0^2}{\Lambda^2}\frac{T}{T_c}\frac{1}{d-4}\right)^{-1} & \text{, for } d > 4 \end{cases} \tag{105}$$

$$A_c = \begin{cases} \frac{\Delta}{M_0^2}\frac{1}{(8\pi)^{d/2}} & \text{, for } d < 2 \\ \frac{\Delta}{8\pi M_0^2} & \text{, for } d = 2 \\ \sin\left[\left(\frac{d}{2}-1\right)\pi\right]\left(\frac{1}{g}+\frac{\Delta M_0^2}{T_C}\right)\frac{(8\pi)^{d/2-1}}{T_C}(d-2) & \text{, for } 2 < d < 4 \\ \frac{1}{2T_C}\left(\frac{1}{2g}+\frac{\Delta M_0^2}{2T_C}\right) & \text{, for } d = 4 \\ \left(\frac{1}{2g}+\frac{\Delta M_0^2}{2T_C}\right)\left(\frac{1}{2g}+\frac{M_0^2}{\Lambda^2}\frac{1}{d-4}\right)^{-1} & \text{, for } d > 4 \end{cases} \tag{106}$$

and

$$A_b = \frac{T_F + g\Delta M_0^2}{(8\pi)^{\frac{d}{2}} g\left[M_0^2\left(1-\frac{T_F}{T_C}\right)\right]^2}. \tag{107}$$

REFERENCES

[1] For a review see A.J.Bray, Adv.Phys. **43**, 357 (1994).

[2] A.Barrat, Phys.Rev. E**57**, 3629 (1998).

[3] G.Parisi, F.Ricci-Terzenghi and J.J.Ruiz-Lorenzo, Eur.Phys.J. B**11**, 317 (1999).

[4] F.Corberi, E.Lippiello and M.Zannetti, Phys.Rev. E **63**, 061506 (2001); cond-mat/0110651.

[5] L.F.Cugliandolo and D.S.Dean, J.Phys.A: Math.Gen. **28**, 4213 (1995); W.Zippold,R.Kuhn and H.Horner, cond-mat/9904329.

[6] L.Berthier, J.L.Barrat and J.Kurchan, Eur.Phys.J. B**11**, 635 (1999)

[7] E.Lippiello and M.Zannetti, Phys.Rev. E**61**, 3369 (2000).

[8] G.F.Mazenko, O.T.Valls and M.Zannetti, Phys.Rev. B**38**, 520 (1988); S.Franz and M.A.Virasoro, J.Phys.A: Math.Gen. **33**, 891 (2000).

[9] C.Godreche and J.M.Luck, J.Phys.A: Math.Gen. **33**, 1151 (2000).

[10] S.Franz, M.Mezard, G.Parisi and L.Peliti, Phys.Rev.Lett. **81**, 1758 (1998); J.Stat.Phys. **97**, 459 (1998).

[11] A.Coniglio and M.Zannetti, Europhys. Lett. **10**, 575 (1989).

[12] C.Godreche and J.M.Luck, J.Phys.A: Math.Gen. **33**, 9141 (2000).

[13] R.J.Baxter *Exactly Solved Models in Statistical Mechanics*, Academic Press 1982.

[14] C.Castellano, F.Corberi and M.Zannetti, Phys.Rev. E**56**, 4973 (1997).

[15] T.J.Newman and A.J.Bray, J.Phys.A: Math.Gen. **23**, 4491 (1990).

FIGURES

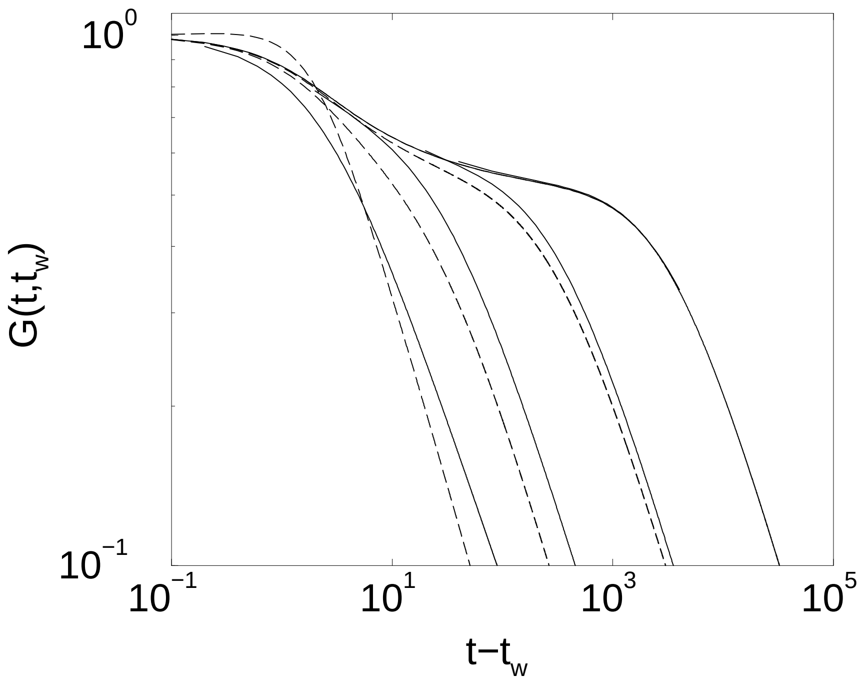

FIG. 1. Comparison of the exact autocorrelation function $G(t,t_w)$ (broken line) with the sum $G_{st}(t-t_w)+G_{ag}(t/t_w)$ (continous line) for $t_w=1,10,10^2,10^3$ increasing from left to right. For $t_w=10^3$ the two curves are indistinguishable. Parameters of the quench are $d=3, T_F=T_C/2$ and $\Delta=1$.

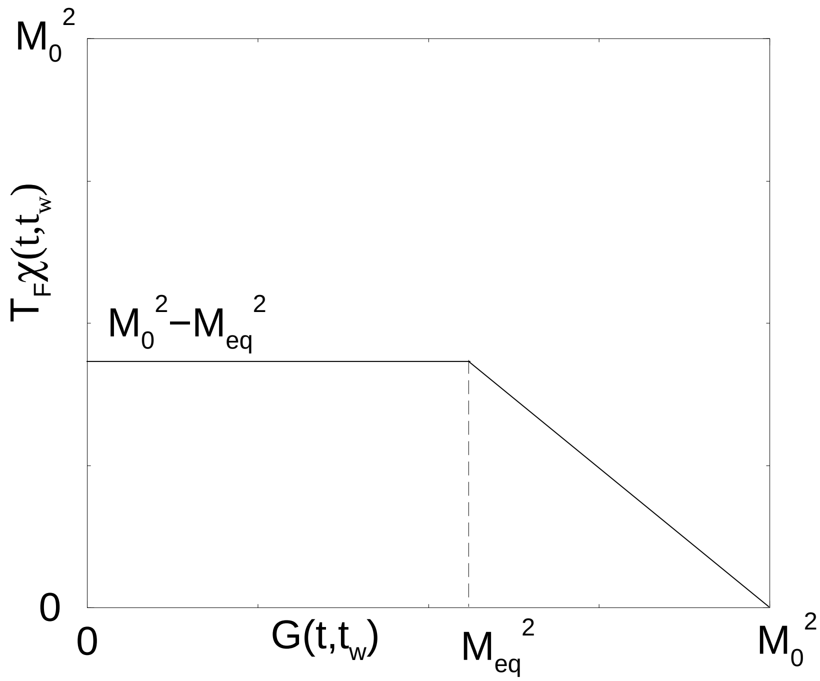

FIG. 2. Parametric plot of $T_F\chi(t,t_w)$ vs. $G(t,t_w)$ in the limit $t_w \to \infty$ for $d > 2$.

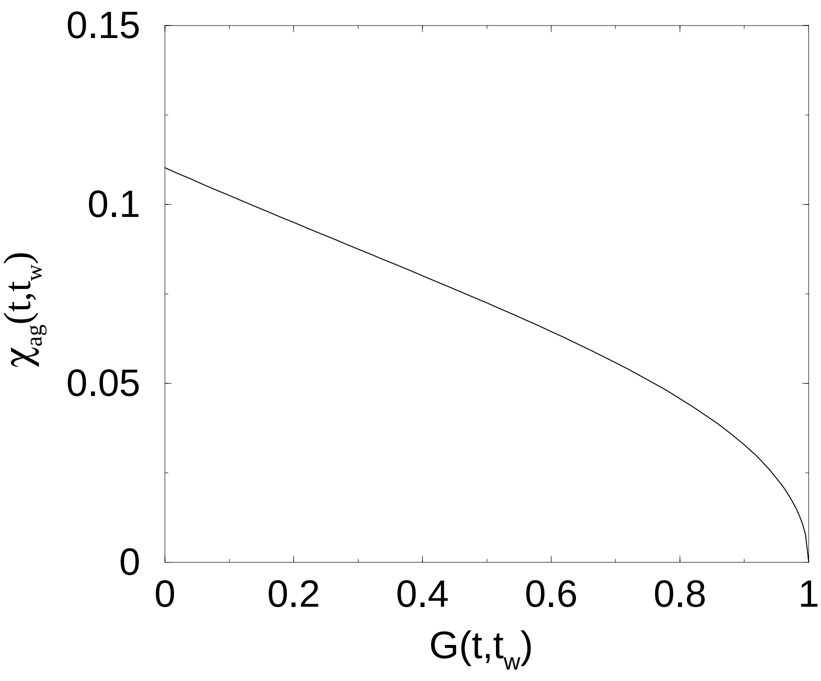

FIG. 3. Parametric plot of $\chi_{ag}(t,t_w)$ vs. $G(t,t_w)$ for $d = 2$

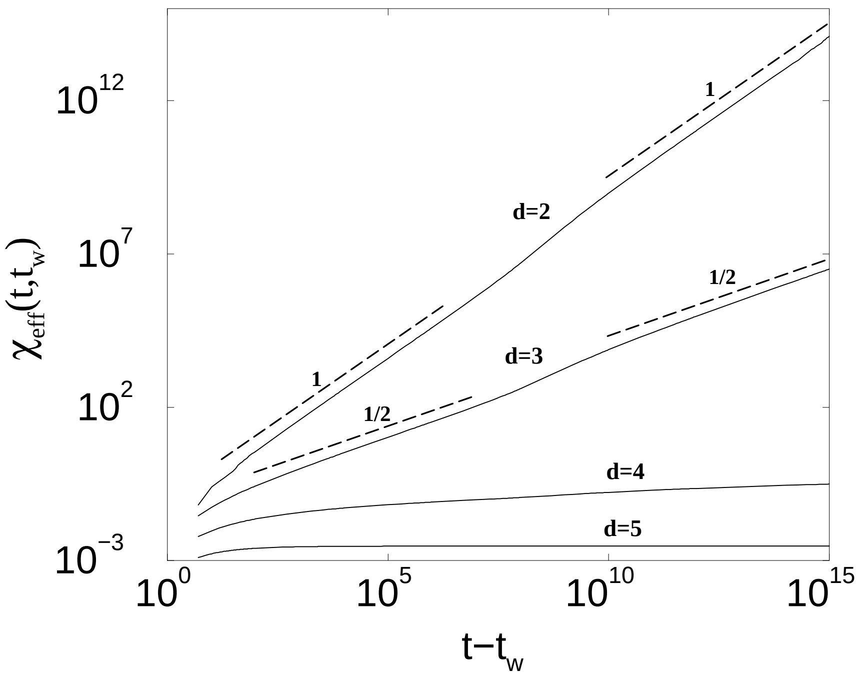

FIG. 4. Plot of $\chi_{eff}(t, t_w)$ for $t_w = 10^8$ and $\Lambda = 1$. The dashed lines are power laws with the corresponding exponent α.